Name: ________________________________

1. Quantum Mechanics

Consider a 1D harmonic oscillator with creation and annihilation operators $a^\dagger$ and a with the usual commutation relation $[a, a^\dagger] = 1$. An eigenstate of the annihilation operator is called a "coherent state," and is denoted with its eigenvalue β, which is in general a complex number:

$$a|\beta\rangle = \beta|\beta\rangle.$$

(a) (14 points) Derive the expansion for a coherent state $|\beta\rangle$ in the basis of number (Fock) states $|n\rangle$ where n is a non-negative integer. Choose a global phase such that $\langle n = 0|\beta\rangle$ is a positive, real number. Hint: Recall that $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

(b) (6 points) Calculate the overlap integral between two coherent states, $\langle\beta|\alpha\rangle$. Are coherent states with $\alpha \neq \beta$ ever strictly orthogonal?

2. Quantum Mechanics

Here we study a simple one-dimensional quantum-mechanical system that resembles the hydrogen molecular ion. A particle of mass m is in a potential

$V(x) = -A_0 [\ \delta(x-a) + \delta(x+a)\ ]$, where $A_0 > 0$.

(a) (3 points) The eigenstates of the (non-relativistic) Hamiltonian for this potential will have definite parity. Without doing any calculation, sketch the amplitude of an even eigenstate as a function of x.

(b) (7 points) Find the expression that determines the bound-state energy for even-parity states, and determine graphically how many even-parity bound states exist.

(c) (10 points) Repeat parts (a) and (b) for odd parity. For what values of A_0 is there at least one such bound state?

Hints: If you take advantage of the symmetry of the problem to minimize the number of constants-to-be determined in your eigenfunctions (and you should), the constraints on the eigenfunctions at $x = a$ and $x = -a$ will be redundant. You can solve the problem without explicitly normalizing the eigenfunctions.

Name: ________________________________

3. Quantum Mechanics

Consider a spin-half electron in a p-state ($\ell = 1$). The total angular momentum is $\mathbf{J}=\mathbf{L}+\mathbf{s}$. Add explicitly the angular momenta and find the quartet and the doublet in the combined representation in terms of the original representation.

Hint: For any operators satisfying angular momentum algebra, the raising and the lowering operators satisfy

$$J_{\pm}\,|j,m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|j, m \pm 1\rangle\,,$$

where m is the corresponding z-component, ($\hbar = 1$). Once you have worked out the quartet states you should be able to write down the doublet states by inspection.

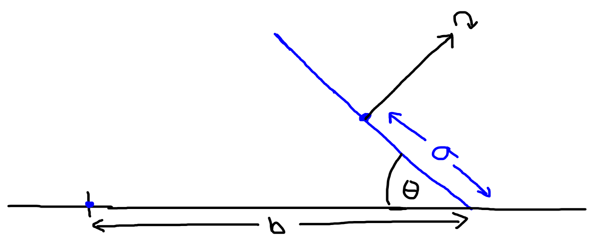
4. Quantum Mechanics

A one-dimensional harmonic oscillator of charge e is perturbed by an electric field E in the positive x-direction. Calculate the change in each energy level to second order in the perturbation, and calculate the induced electric dipole moment. Show that the problem can be solved exactly, and compare the result with perturbation approximation.

Hint: $a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$ and $a|n\rangle = \sqrt{n}\,|n-1\rangle$ acting on harmonic oscillator states $|n\rangle$, where $a^\dagger$ and a are the creation and the destruction operators. Note also the definitions: $a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$ and $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$.

5. Classical Mechanics

A coin, idealized as a uniform disk of radius a and negligible thickness, is rolling in a circle on a table. The point of contact describes a circle of radius b on the table. The plane of the coin makes an angle θ with the plane of the table. Find the angular velocity ω of the motion of the center of mass of the coin. Hint: you don't need to use a Lagrangian for this problem, just Newton's laws.



Name: ______________________________

6. Classical Mechanics

Consider a pendulum consisting of a point mass m attached to a string of slowly increasing length $l(t)$. The motion is confined to a plane and we assume that $|l/\dot{l}|$ is much greater than the period of oscillation.

(a) (6 points) Find the Lagrangian $L(\theta, \dot{\theta}, t)$ and the Hamiltonian $H(\theta, p_\theta, t)$ of the system (θ is the angle of oscillation and p_θ is the conjugate momentum).

(b) (4 points) Is the Hamiltonian H equal to the total energy E of the pendulum? Are E and H conserved?

(c) (5 points) Derive the equation of motion for θ in the form of a second order ordinary differential equation. When $\dot{l} = 0$, what is the frequency of small oscillations?

(d) (5 points) Show that the amplitude of small oscillations is proportional to $l^{-3/4}$ as the length of the string $l(t)$ varies. (Hint: Consider using $\int d\theta p_\theta$ as an adiabatic invariant.)

Physics - Comprehensive Exam Name: ________________________________
Fall 2017

1. Electromagnetism

An infinite straight wire carrying a current I is suspended parallel to the plane interface between vacuum and a medium with magnetic permeability $\mu \neq 1$, at a distance a from the interface. Calculate the force per unit length on the wire, and state whether it is attractive or repulsive.

Hint: for this problem, it is helpful to introduce a scalar potential. Also, it is helpful to take coordinates in the complex plane perpendicular to the wire.

2. Electromagnetism

The possibility that photons have some small, nonzero rest mass m can be introduced consistently into Maxwell's equations to transform them into the Proca equations:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} - \frac{m^2 c^2}{\hbar^2}\Phi$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} - \frac{m^2 c^2}{\hbar^2}\boldsymbol{A},$$

which are valid for the Lorentz gauge, with the familiar relations given by

$$\nabla \cdot \boldsymbol{A} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial t}$$
$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$
$$\boldsymbol{E} = -\nabla\Phi - \frac{\partial \boldsymbol{A}}{\partial t}$$
$$c \equiv \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

(a) (13 points) Derive the modified wave equation obeyed by $\boldsymbol{E}$ in free space (*i.e.* no charge or current density).

(b) (7 points) Consider transverse electromagnetic plane waves that propagate through free space along the x-axis with electric field magnitude

$$E(x,t) = \frac{1}{2}\mathcal{E}_0\left(e^{i(kx-\omega t)} + e^{-i(kx-\omega t)}\right)$$

Derive the expression for the group velocity as a function of optical frequency $v_g(\omega) \equiv \frac{d\omega}{dk}$ of electromagnetic waves in free space.

Potentially useful vector math: $\nabla \times (\nabla \times \boldsymbol{G}) = \nabla(\nabla \cdot \boldsymbol{G}) - \nabla^2 \boldsymbol{G}$.

3. Electromagnetism

(a) (5 points) An element of wire of oriented length $d\boldsymbol{\ell}$ is moving with velocity $\boldsymbol{v}$ in a magnetic field $\boldsymbol{B}$. Starting from the Lorentz force law, calculate the motional EMF $d\varepsilon$ developed in the element.

(b) (15 points) A large sheet of copper moves with constant velocity $\boldsymbol{v}$ through the narrow gap of a C-shaped permanent magnet. The copper has thickness h and conductivity σ. The magnet's field may be considered to have a constant value B_0 inside, and be negligible outside, the rectangular area $w\times\ell$ determined by the magnet's pole pieces (take the sheet's velocity to be parallel to the w dimension). The motion induces an EMF in the conducting sheet, which drives a two-dimensional pattern of eddy currents in the sheet. The portion of this current pattern flowing within the region of magnetic field experiences a Lorentz force. Calculate, to within a constant of proportionality α, the resulting electromagnetic force on the moving sheet (state explicitly the direction of this force).

The dimensionless constant α is of order unity and is meant to save you the trouble of calculating the exact path of the eddy currents in the sheet. Take α to be the constant of proportionality relating the total resistance of the current path (which is difficult to calculate) to the resistance of a piece of the current path that you can calculate easily. However you decide to define α, be sure to state your definition clearly.

4. Electromagnetism

Consider the backscattering of laser photons from a counter-propagating relativistic electron. The electron is taken to be traveling in the z-direction with speed v giving a Lorentz facton $\gamma = \left[1-(v/c)^2\right]^{-1/2}$. The laser photons propagate in the opposite (-z) direction and the scattered photons in the positive z-direction. The laser photons have a free-space wavelength of 800 nm ($\hbar\omega = 1.55$ eV), and the electron total energy is 100 MeV.

(a) Assuming the *Thomson* approximation, the frequency ω' of the scattered radiation in the *electron rest frame* obeys $\hbar\omega' << m_e c^2$, and the backscattered radiation has nearly the same frequency, but opposite wavenumber k' (reversed propagation direction) as the laser in this frame. Write expressions for k' and ω', and evaluate the adequacy of the Thomson approximation.

(b) What is the energy of the scattered photons in the laboratory frame?

5. Statistical Mechanics

Consider a closed LC circuit. It is to be used as a thermometer, by measuring the rms voltage across the capacitance (and inductance). Find an expression for the temperature dependence of the rms voltage, valid for all temperatures. Then find the limits for high and low temperature.

Hint: at low enough temperatures quantum effects may be important.

Name: ________________________________

6. Statistical Mechanics

(a) (4 points) Consider a polymer chain comprising N segments. Each segment of length d can freely rotate relative to each other. The system is in 3D and at temperature T . What is the mean displacement $\langle \mathrm{r}_{1N} \rangle$ between the chain ends?

(b) (8 points) Apply a stretching force f to both ends of the polymer. Find the partition function as a function of f and then use it to derive the mean distance between the ends.

(c) (8 points) Now, the polymer in (a) is suspended at one end in a gravitational field of strength g. The mass of each segment of the chain is m. Calculate the average length of the chain. (Hint: you can replace a sum by an integral when N is large.)